

Liouville theorems for scaling invariant superlinear parabolic problems with gradient structure *

Pavol Quittner

Department of Applied Mathematics and Statistics, Comenius University
Mlynská dolina, 84248 Bratislava, Slovakia
`quittner@fmph.uniba.sk`

Abstract

We provide a simple method for obtaining new Liouville theorems for scaling invariant superlinear parabolic problems with gradient structure. To illustrate the method we prove Liouville theorems (guaranteeing nonexistence of positive classical solutions) for the following model problems: the scalar nonlinear heat equation

$$u_t - \Delta u = u^p \quad \text{in } \mathbb{R}^n \times \mathbb{R},$$

its vector-valued generalization with a p -homogeneous nonlinearity and the linear heat equation in $\mathbb{R}_+^n \times \mathbb{R}$ complemented by nonlinear boundary conditions of the form $\partial u / \partial \nu = u^q$. Here ν denotes the outer unit normal on the boundary of the halfspace \mathbb{R}_+^n and the exponents $p, q > 1$ satisfy $p < n/(n-2)$ and $q < (n-1)/(n-2)$ if $n > 2$ (or $p < (n+2)/(n-2)$ and $q < n/(n-2)$ if $n = 2$ and some symmetry of the solutions is assumed). As a typical application of our nonexistence results we provide optimal universal estimates for positive solutions of related problems in bounded and unbounded domains.

1 Introduction

In this paper we consider several model scaling invariant parabolic problems with gradient structure and prove that these problems — in a certain range of given parameters — do not possess positive entire solutions, i.e. solutions defined for all times $t \in (-\infty, +\infty)$. Such a result will be called (parabolic) Liouville theorem.

Nonlinear heat equation. Let us first consider the scalar nonlinear heat equation

$$u_t - \Delta u = u^p, \quad x \in \mathbb{R}^n, \quad t \in \mathbb{R}, \quad (1)$$

*Supported in part by the Slovak Research and Development Agency under the contract No. APVV-0134-10 and by VEGA grant 1/0711/12.

where $p > 1$, $n \geq 1$ and $u = u(x, t) > 0$. Since problem (1) possesses positive stationary solutions if $n > 2$ and $p \geq (n + 2)/(n - 2)$, the necessary condition for the Liouville theorem for (1) is $p < (n + 2)/(n - 2)_+$. This condition is also sufficient if we restrict ourselves to radially symmetric solutions, see [22, 24]. In the general non-radial case, the Liouville theorem for (1) was proved in [2] only under the assumption $n = 1$ or $n > 1$ and $p < n(n + 2)/(n - 1)^2$. In particular, if $n = 2$ then one has to assume $p < 8$. Our main result for problem (1) guarantees that for $n = 2$ this assumption on p is superfluous. More precisely, we prove the following Liouville theorem.

Theorem 1. *Let $p > 1$, $(n - 2)p < n$. Then the equation (1) does not possess positive classical solutions.*

If $n > 2$ then $n/(n - 2) < n(n + 2)/(n - 1)^2$ so that the assertion in Theorem 1 follows from [2] whenever $n \neq 2$. We formulate and prove our result for general n since our method is very different from that in [2] and it can also be used for problems where the arguments of [2] cannot be used or have not been used so far. In particular, in this paper we also consider a vector-valued generalization of (1) and the linear heat equation complemented by nonlinear boundary conditions and in these cases we obtain new results for all $n \geq 1$. It should be emphasized that we do not exploit the semilinear structure of our problems: we consider these model problems just for simplicity. Our method is based on scaling and energy estimates for the rescaled problems. This approach enables us to show that any positive bounded entire solution of the parabolic problem has to be time-independent so that the nonexistence result for bounded solutions follows from the corresponding elliptic Liouville theorem (and then the nonexistence of unbounded solutions is often an easy consequence of doubling and scaling arguments). Let us note that if $n > 2$ and $p > (n + 2)/(n - 2)$ then, in addition to positive bounded stationary solutions, there also exist positive bounded entire solutions of (1) which do depend on time; in particular there exist homoclinic solutions, see [9].

Liouville theorems have important consequences concerning universal a priori estimates for positive solutions of related problems. To be more specific, let us formulate a typical result of this type based on Theorem 1. Since our result in Theorem 1 is new only if $n = 2$, we restrict ourselves to this case. Consider nonnegative solutions of the equation

$$u_t - \Delta u = f(u) \quad \text{in } \Omega \times (T_1, T_2), \quad (2)$$

where $f : [0, \infty) \rightarrow \mathbb{R}$ is a continuous function satisfying

$$\lim_{u \rightarrow +\infty} u^{-p} f(u) = \ell \in (0, \infty), \quad (3)$$

and

$$\Omega \text{ is an arbitrary domain in } \mathbb{R}^2, \quad -\infty \leq T_1 < T_2 \leq \infty. \quad (4)$$

The following theorem is a direct consequence of Theorem 1 and (the proof of) [24, Theorems 3.1 and 4.1]; cf. also [24, Remark 3.4(e)].

Theorem 2. Assume $p > 1$, (3), (4) and let u be a nonnegative classical solution of (2). Then

$$u(x, t) \leq C(C_1 + (t - T_1)^{-\beta} + (T_2 - t)^{-\beta} + C_2 \text{dist}^{-2\beta}(x, \partial\Omega)) \quad \text{in } \Omega \times (T_1, T_2), \quad (5)$$

where $\beta := 1/(p - 1)$, $C = C(f) > 0$ is independent of Ω , T_1 , T_2 and u , $C_1 = 0$ if $f(u) = u^p$, $C_1 = 1$ otherwise, $C_2 = 1$, $(t - T_1)^{-\beta} := 0$ if $T_1 = -\infty$, $(T_2 - t)^{-\beta} := 0$ if $T_2 = \infty$ and $\text{dist}^{-2\beta}(x, \partial\Omega) := 0$ if $\Omega = \mathbb{R}^2$.

If, in addition, Ω is (uniformly C^2) smooth and u satisfies the boundary condition

$$u = 0 \quad \text{on } \partial\Omega \times (T_1, T_2) \quad (6)$$

then (5) is true with $C = C(f, \Omega)$, $C_1 = 1$ and $C_2 = 0$.

In particular, if $\Omega \subset \mathbb{R}^2$ is smooth and u is any positive solution of the problem (2),(6) which blows up at $t = T_2$ then Theorem 2 guarantees that the blow-up rate is of type I and the corresponding estimate is universal (i.e. the constant C in (5) does not depend on u).

Another application of Theorem 2 deals with so called ancient solutions. Assume $T \in \mathbb{R}$, $1 < p$ and $(n - 2)p < n + 2$. Then [19, Corollary 1.6] gives a complete characterization of all (positive classical) solutions of the problem

$$u_t - \Delta u = u^p, \quad x \in \mathbb{R}^n, \quad t \in (-\infty, T), \quad (7)$$

under the assumption

$$u(x, t) \leq C(T - t)^{-\beta}. \quad (8)$$

Theorem 2 guarantees that (8) is always true if $n = 2$. In fact, the assertions in Theorem 2 (hence also (8)) are true for any n and $p > 1$ such that (1) does not possess positive classical solutions.

Vector valued case. Our next model problem is a vector-valued generalization of (1): we consider positive classical solutions $U = (u_1, u_2, \dots, u_m)$ of the system

$$U_t - \Delta U = F(U), \quad x \in \mathbb{R}^n, \quad t \in \mathbb{R}, \quad (9)$$

where

$$F = \nabla G, \quad \text{with } G \in C_{loc}^{2+\alpha}(\mathbb{R}^m, \mathbb{R}) \text{ for some } \alpha > 0, \quad (10)$$

$$G(0) = 0, \quad G(U) > 0 \quad \text{for } U \neq 0, \quad (11)$$

$$F(\lambda U) = \lambda^p F(U) \quad \text{for } U \in \mathbb{R}^m, \quad \lambda > 0, \quad (12)$$

$$\text{there exists } \xi \in (0, \infty)^m \text{ such that } \xi \cdot F(U) > 0 \quad \text{for } U \neq 0. \quad (13)$$

Using the same arguments as in the proof of Theorem 1 we prove the following theorem.

Theorem 3. Assume (10), (11), (12), (13) and $p > 1$, $(n - 2)p < n$. Then the system (9) does not possess nontrivial nonnegative classical solutions.

Notice that Theorem 1 is a special case of Theorem 3. We will first prove Theorem 1 (in order to explain the idea of our method by using the simplest possible model problem); the proof of Theorem 3 will then follow the proof of Theorem 1.

Theorem 3 for $n = 1$ and the approach in [1, Proposition 2.4] (see also [26] and [21]) enable us to prove also the following theorem.

Theorem 4. *Assume (10), (11), (12), (13) and $p > 1$, $(n - 2)p < n + 2$. Then the system (9) does not possess nontrivial nonnegative classical radially symmetric solutions.*

Theorem 4 is a generalization of the scalar parabolic Liouville theorem for radially symmetric solutions of (1) proved in [22, 24] by completely different arguments. Similarly as in the scalar case, Theorems 3 and 4 can be used in order to prove universal a priori estimates of positive solutions of many related problems.

As far as we know, if $n, m > 1$ then the only known nonexistence results for (9) in the non-radial case are of Fujita-type and require the strong condition $p \leq (n + 2)/n$. If $n = 1$, $m = 2$ and

$$F(u_1, u_2) = (u_1^p - \lambda u_1^r u_2^{r+1}, u_2^p - \lambda u_1^{r+1} u_2^r), \quad p = 2r + 1 > 1, \quad (14)$$

then by using the approach in [2], a Liouville theorem for (9) has very recently been established in [21] under the assumption $\lambda < r/(3r + 2)$. Notice that in this particular case, Theorem 3 guarantees the nonexistence for any $\lambda < 1$ and this condition on λ is optimal.

In the radial setting, assuming $m = 2$, (14) and either $p = 3 \geq n$, $\lambda < 1$ or $p(n - 2) < n + 2$, $\lambda < r/(3r + 2)$, nonexistence results for (9) have also been obtained in [26] or [21], respectively.

Nonlinear boundary conditions. Next consider positive classical solutions of the problem

$$\left. \begin{aligned} u_t - \Delta u &= 0 && \text{in } \mathbb{R}_+^n \times \mathbb{R}, \\ u_\nu &= u^q && \text{on } \partial \mathbb{R}_+^n \times \mathbb{R}, \end{aligned} \right\} \quad (15)$$

where $\mathbb{R}_+^n := \{(x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n : x_1 > 0\}$, $\nu = (-1, 0, 0, \dots, 0)$ is the outer unit normal on the boundary $\partial \mathbb{R}_+^n = \{x \in \mathbb{R}^n : x_1 = 0\}$ and $q > 1$. In this case our method yields the following result.

Theorem 5. *Let $q > 1$, $(n - 2)q < n - 1$. Then the problem (15) does not possess positive classical bounded solutions.*

The result in Theorem 5 is new for any $n \geq 1$. If $n = 1$ then this nonexistence result was proved in [26] by other arguments, but only for solutions with bounded spatial derivatives. For general $n \geq 1$ the only known nonexistence results for (15) are of Fujita-type and require $q \leq (n + 1)/n$, see [10, 5].

Liouville theorem for stationary solutions of (15) is true for $q < n/(n - 2)_+$ (see [15]) and this condition on q is optimal: if $n > 2$ and $q = n/(n - 2)$ then there exists a stationary solution of (15) of the form $u(x) = c|x - x_0|^{2-n}$, where the first component of

x_0 is negative (see [13] and the references therein for the analysis of stationary solutions for $q \geq n/(n-2)$). Under the optimal assumption $q < n/(n-2)_+$ we can also prove nonexistence of solutions of (15) exhibiting the following axial symmetry:

$$u(x_1, \tilde{x}, t) = v(x_1, |\tilde{x}|, t), \quad \text{where } \tilde{x} = (x_2, x_3, \dots, x_n). \quad (16)$$

Theorem 6. *Let $q > 1$, $(n-2)q < n$. Then the problem (15) does not possess positive classical bounded solutions exhibiting the symmetry property (16).*

Theorem 6 is an analogue to Theorem 4 and is proved by similar but technically more advanced arguments.

Let us also mention that the boundedness assumptions in Theorems 5 and 6 still allow applications based on doubling and scaling arguments and yielding a priori estimates for positive solutions of related problems. In particular, Theorem 5 can be used to obtain blow-up rate estimates for positive solutions of the problem

$$\left. \begin{aligned} u_t - \Delta u &= 0 & x \in \Omega, \quad t \in (0, T), \\ u_\nu &= u^q & x \in \partial\Omega, \quad t \in (0, T), \end{aligned} \right\} \quad (17)$$

where ν denotes the outer unit normal on the boundary $\partial\Omega$. More precisely, we will prove the following theorem.

Theorem 7. *Assume that $\Omega \subset \mathbb{R}^n$ is bounded and smooth, $q > 1$, $(n-2)q < n-1$. Assume also that u is a positive classical solution of (17) which blows up at $t = T$. Then there exists $C = C(u) > 0$ such that u satisfies the blow-up rate estimate*

$$u(x, t)(T-t)^{1/2(q-1)} + |\nabla u(x, t)|(T-t)^{q/2(q-1)} \leq C \quad (18)$$

for all $x \in \overline{\Omega}$ and $t \in (T/2, T)$.

If Ω is bounded and $q > 1$ then any positive solution of (17) blows up in finite time. Theorem 7 guarantees that for $1 < q < (n-1)/(n-2)_+$, the blow-up of such solution is of type I, i.e. u satisfies the estimate

$$\|u(\cdot, t)\|_\infty \leq C(T-t)^{-1/2(q-1)} \quad \text{for } t \in (T/2, T). \quad (19)$$

This result for bounded domains was known only under the stronger assumption $1 < q \leq 1 + 1/n$ (see [16]). On the other hand, type I blow-up for both positive and sign-changing solutions of (17) has been established in the full subcritical range $1 < q < n/(n-2)_+$ if Ω is a half-space (see [3] and [27]) and it has also been proved for bounded domains and $1 < q \leq n/(n-2)_+$ in the class of positive, time increasing solutions (see [16]). Let us also mention that the blow-up rate estimate (19) is optimal (see the lower estimates in [17, 18]) and that the blow-up need not be of type I for (some) supercritical q (see [14]).

2 Proof of Theorems 1, 3 and 4

In the proofs we will often need the following lemma.

Doubling Lemma. (see [24, Lemma 5.1]). *Let (X, d) be a complete metric space and $\emptyset \neq D \subset X$. Let $M : D \rightarrow (0, \infty)$ be bounded on compact subsets of D and fix a real $k > 0$. If $y \in D$ is such that*

$$M(y) \operatorname{dist}(y, X \setminus D) > 2k, \quad (20)$$

then there exists $x \in D$ such that

$$M(x) \operatorname{dist}(x, X \setminus D) > 2k, \quad M(x) \geq M(y), \quad (21)$$

and

$$M(z) \leq 2M(x) \quad \text{whenever} \quad \operatorname{dist}(z, x) \leq \frac{k}{M(x)}. \quad (22)$$

Notice that the inequalities in (21) and (22) guarantee

$$\operatorname{dist}(z, x) \leq \frac{k}{M(x)} < \frac{1}{2} \operatorname{dist}(x, X \setminus D),$$

so that $z \in D$ and the value $M(z)$ is well defined. Notice also that if $D = X$ then $\operatorname{dist}(y, X \setminus D) = \operatorname{dist}(y, \emptyset) = \infty$ so that the assumption (20) is satisfied for any $y \in D$. In most cases, we will use the Doubling Lemma with X being a closed subset of $\mathbb{R}^n \times \mathbb{R}$ equipped with the parabolic distance

$$\operatorname{dist}_P((x, t), (\tilde{x}, \tilde{t})) := |x - \tilde{x}| + \sqrt{|t - \tilde{t}|}.$$

Proof of Theorem 1. Assume on the contrary that there exists a positive solution u of (1). Doubling and scaling arguments in [24] guarantee that we may assume that

$$u(x, t) \leq 1 \quad \text{for all } x \in \mathbb{R}^n, t \in \mathbb{R}. \quad (23)$$

In fact, assume that $u(x_0, t_0) > 1$ for some $(x_0, t_0) \in \mathbb{R}^n \times \mathbb{R}$. For any $k = 1, 2, \dots$, the Doubling Lemma (with $D = X = \mathbb{R}^n \times \mathbb{R}$, $\operatorname{dist} = \operatorname{dist}_P$, $M = u^{(p-1)/2}$ and $y = (x_0, t_0)$) guarantees the existence of (x_k, t_k) such that

$$\begin{aligned} M_k &:= u^{(p-1)/2}(x_k, t_k) \geq u^{(p-1)/2}(x_0, t_0) \quad \text{and} \\ u^{(p-1)/2}(x, t) &\leq 2M_k \quad \text{whenever} \quad |x - x_k| + \sqrt{|t - t_k|} \leq \frac{k}{M_k}. \end{aligned}$$

The rescaled functions

$$v_k(y, s) := \lambda_k^{2/(p-1)} u(x_k + \lambda_k y, t_k + \lambda_k^2 s), \quad \text{where} \quad \lambda_k = \frac{1}{2M_k},$$

are positive solutions of (1) and satisfy $v_k(0, 0) = 2^{-2/(p-1)}$, $v_k(y, s) \leq 1$ for $|y| + \sqrt{|s|} \leq 2k$. The parabolic regularity guarantees that the sequence $\{v_k\}$ is relatively compact (in C_{loc} , for example), so that a suitable subsequence of $\{v_k\}$ converges to a nonnegative solution

v of (1) satisfying $v \leq 1$. Since $v(0, 0) > 0$, we have $v > 0$ by the maximum principle and uniqueness. Consequently, replacing u by v we may assume that (23) is true.

Denote $c_0 := u(0, 0)$ and $\beta := 1/(p-1)$. For $y \in \mathbb{R}^n$, $s \in \mathbb{R}$ and $k = 1, 2, \dots$ set

$$w_k(y, s) := (k-t)^\beta u(y, \sqrt{k-t}, t), \quad \text{where } s = -\log(k-t), \quad t < k.$$

Set also $s_k := -\log k$ and notice that $w = w_k$ solve the problem

$$\left. \begin{aligned} w_s &= \Delta w - \frac{1}{2}y \cdot \nabla w - \beta w + w^p \\ &= \frac{1}{\rho} \nabla \cdot (\rho \nabla w) - \beta w + w^p \quad \text{in } \mathbb{R}^n \times \mathbb{R}, \end{aligned} \right\} \quad (24)$$

where $\rho(y) := e^{-|y|^2/4}$. In addition,

$$w_k(0, s_k) = k^\beta c_0$$

and

$$\|w_k(\cdot, s)\|_\infty \leq e^{2\beta} k^\beta \quad \text{for } s \in [s_k - 2, \infty). \quad (25)$$

Set

$$E_k(s) := \frac{1}{2} \int_{\mathbb{R}^n} (|\nabla w_k(y, s)|^2 + \beta w_k^2(y, s)) \rho(y) dy - \frac{1}{p+1} \int_{\mathbb{R}^n} w_k^{p+1}(y, s) \rho(y) dy.$$

Then in the same way as in [12, (2.25) and Proposition 2.1] we obtain $E_k(s) \geq 0$ and, given $\sigma < s_k$,

$$\left. \begin{aligned} &\frac{1}{2} \left(\int_{\mathbb{R}^n} w_k^2(y, s_k) \rho(y) dy - \int_{\mathbb{R}^n} w_k^2(y, \sigma) \rho(y) dy \right) \\ &= -2 \int_\sigma^{s_k} E_k(s) ds + \frac{p-1}{p+1} \int_\sigma^{s_k} \int_{\mathbb{R}^n} w_k^{p+1}(y, s) \rho(y) dy ds, \end{aligned} \right\} \quad (26)$$

$$\int_\sigma^{s_k} \int_{\mathbb{R}^n} \left| \frac{\partial w_k}{\partial s}(y, s) \right|^2 \rho(y) dy ds = E_k(\sigma) - E_k(s_k) \leq E_k(\sigma). \quad (27)$$

Multiplying equation (24) by ρ , integrating over $y \in \mathbb{R}^n$ and using Jensen's inequality yields

$$\begin{aligned} &\frac{d}{ds} \int_{\mathbb{R}^n} w_k(y, s) \rho(y) dy + \beta \int_{\mathbb{R}^n} w_k(y, s) \rho(y) dy \\ &= \int_{\mathbb{R}^n} w_k^p(y, s) \rho(y) dy \geq C_{n,p} \left(\int_{\mathbb{R}^n} w_k(y, s) \rho(y) dy \right)^p, \end{aligned}$$

where $C_{n,p} := (4\pi)^{-n(p-1)/2}$, which (as in the proof of [8, Theorem 1], for example) implies the estimates

$$\int_{\mathbb{R}^n} w_k(y, s) \rho(y) dy \leq \tilde{C}_{n,p} \quad (28)$$

and

$$\int_\sigma^{s_k} \int_{\mathbb{R}^n} w_k^p(y, s) \rho(y) dy ds \leq \tilde{C}_{n,p} (1 + \beta(s_k - \sigma)), \quad (29)$$

where $\tilde{C}_{n,p} = (\beta/C_{n,p})^\beta$. The monotonicity of E_k , (26), (25), (28) and (29) guarantee

$$\begin{aligned}
2E_k(s_k - 1) &\leq 2 \int_{s_k-2}^{s_k-1} E_k(s) ds \leq 2 \int_{s_k-2}^{s_k} E_k(s) ds \\
&\leq \frac{1}{2} \int_{\mathbb{R}^n} w_k^2(y, s_k - 2) \rho(y) dy + \frac{p-1}{p+1} \int_{s_k-2}^{s_k} \int_{\mathbb{R}^n} w_k^{p+1}(y, s) \rho(y) dy ds \\
&\leq e^{2\beta} k^\beta \left(\int_{\mathbb{R}^n} w_k(y, s_k - 2) \rho(y) dy + \int_{s_k-2}^{s_k} \int_{\mathbb{R}^n} w_k^p(y, s) \rho(y) dy ds \right) \\
&\leq 2C(n, p) k^\beta,
\end{aligned}$$

where $C(n, p) := e^{2\beta} \tilde{C}_{n,p} (1 + \beta)$, hence $E_k(s_k - 1) \leq C(n, p) k^\beta$. This estimate and (27) guarantee

$$\int_{s_k-1}^{s_k} \int_{\mathbb{R}^n} \left| \frac{\partial w_k}{\partial s}(y, s) \right|^2 \rho(y) dy ds \leq C(n, p) k^\beta. \quad (30)$$

Denote $\lambda_k := k^{-1/2}$ and set

$$v_k(z, \tau) := \lambda_k^{2/(p-1)} w_k(\lambda_k z, \lambda_k^2 \tau + s_k), \quad z \in \mathbb{R}^n, \quad -k \leq \tau \leq 0.$$

Then $0 < v_k \leq e^{2\beta}$, $v_k(0, 0) = c_0$,

$$\frac{\partial v_k}{\partial \tau} - \Delta v_k - v_k^p = -\lambda_k^2 \left(\frac{1}{2} z \cdot \nabla v_k + \beta v_k \right)$$

and, denoting $\alpha := -n + 2 + 4/(p-1)$ and using (30) we also have

$$\begin{aligned}
\int_{-k}^0 \int_{|z| < \sqrt{k}} \left| \frac{\partial v_k}{\partial \tau}(z, \tau) \right|^2 dz d\tau &= \lambda_k^\alpha \int_{s_k-1}^{s_k} \int_{|y| < 1} \left| \frac{\partial w_k}{\partial s}(y, s) \right|^2 dy ds \\
&\leq C(n, p) e^{1/4} k^{-\alpha/2+\beta} \rightarrow 0 \quad \text{as } k \rightarrow \infty.
\end{aligned}$$

Now the same arguments as in [12] show that (up to a subsequence) the sequence $\{v_k\}$ converges to a positive solution $v = v(z)$ of the problem $\Delta v + v^p = 0$ in \mathbb{R}^n , which contradicts the elliptic Liouville theorem in [11].

Notice that the explicit formula

$$v_k(z, \tau) = e^{-\beta\tau/k} u(e^{-\tau/2k} z, k(1 - e^{-\tau/k}))$$

guarantees $v_k \rightarrow u$. Notice also that if $p = n/(n-2)$ and if we rescaled the functions w_k on the intervals $[s_k - 1, s_k + 1]$ instead of $[s_k - 1, s_k]$ then the above arguments would guarantee $\int_{-\infty}^{\infty} \int_{\mathbb{R}^n} u_t^2 dx dt \leq C(n, p) e^{1/4}$. \square

Proof of Theorem 3. Assume on the contrary that there exists a nontrivial nonnegative solution U of (9). As in the proof of Theorem 1, doubling and scaling arguments in [24] guarantee that we may assume

$$|U(x, t)| \leq 1 \quad \text{for all } x \in \mathbb{R}^n, \quad t \in \mathbb{R}.$$

Denote $C_0 := U(0, 0)$ and $\beta := 1/(p-1)$. For $y \in \mathbb{R}^n$, $s \in \mathbb{R}$ and $k = 1, 2, \dots$ set

$$W_k(y, s) := (k-t)^\beta U(y\sqrt{k-t}, t), \quad \text{where } s = -\log(k-t), \quad t < k.$$

Set also $s_k := -\log k$ and notice that $W = W_k$ solve the problem

$$\left. \begin{aligned} W_s &= \Delta W - \frac{1}{2}y \cdot \nabla W - \beta W + F(W) \\ &= \frac{1}{\rho} \nabla \cdot (\rho \nabla W) - \beta W + F(W) \quad \text{in } \mathbb{R}^n \times \mathbb{R}, \end{aligned} \right\} \quad (31)$$

where $\rho(y) = e^{-|y|^2/4}$. In addition,

$$W_k(0, s_k) = k^\beta C_0 \quad \text{and} \quad \|W_k(\cdot, s)\|_\infty \leq e^{2\beta} k^\beta \quad \text{for } s \in [s_k - 2, \infty).$$

Set

$$E_k(s) := \frac{1}{2} \int_{\mathbb{R}^n} (|\nabla W(y, s)|^2 + \beta W^2(y, s)) \rho(y) dy - \int_{\mathbb{R}^n} G(W(y, s)) \rho(y) dy.$$

Since assumptions (10), (11), (12) and (13) guarantee

$$C_G |W|^{p+1} \geq G(W) = \frac{1}{p+1} F(W) W \geq c_G |W|^{p+1}, \quad \xi \cdot F(W) \geq c_F |W|^p,$$

one can use the same arguments as in the proof of Theorem 1 to show that $E_k(s_k - 1) \leq C k^\beta$ for some C depending only on n, p, C_G, c_F and ξ . In fact, to prove the analogues of (28) and (29), for example, it is sufficient to multiply the i -th component in (31) by $\xi_i \rho$, integrate and sum over i . Consequently, as in the proof of Theorem 1 the functions

$$V_k(z, \tau) := \lambda_k^{2/(p-1)} W_k(\lambda_k z, \lambda_k^2 \tau + s_k), \quad z \in \mathbb{R}^n, \quad -k \leq \tau \leq 0$$

converge (up to a subsequence) to a positive solution $V = V(z)$ of the problem $\Delta V + F(W) = 0$ in \mathbb{R}^n , which contradicts the elliptic Liouville theorem [28, Theorem 6(i)]. \square

Proof of Theorem 4. The proof is based on the same arguments as the proof of [1, Proposition 2.4] (cf. also [26, Theorem 4.1]). For the reader's convenience (and since we will also need a nontrivial modification of these arguments in the proof of Theorem 6) we provide a detailed proof.

Let U be a nontrivial nonnegative radial solution of (9). Since U is radial, there exists $\tilde{U} : [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}^m : (r, t) \mapsto \tilde{U}(r, t)$ such that $U(x, t) = \tilde{U}(|x|, t)$.

First we show that we can assume that U is bounded. In fact, assume that there exist $r_k \in [0, \infty)$ and $t_k \in \mathbb{R}$ such that $|\tilde{U}(r_k, t_k)| \rightarrow \infty$. The Doubling Lemma (with $D = X = [0, \infty) \times \mathbb{R}$, $\text{dist} = \text{dist}_P$ and $M = |\tilde{U}|^{(p-1)/2}$) guarantees that we may assume

$$M(r, t) \leq 2M_k \quad \text{whenever} \quad |r - r_k| + \sqrt{|t - t_k|} \leq \frac{k}{M_k},$$

where $M_k := |\tilde{U}(r_k, t_k)|^{(p-1)/2}$. Set $\rho_k := r_k M_k$ and $\lambda_k := 1/M_k$. Passing to a subsequence we may assume $\rho_k \rightarrow \rho_\infty \in [0, \infty]$. If $\rho_\infty = \infty$ then the functions

$$V_k(\rho, s) := \lambda_k^{2/(p-1)} \tilde{U}(r_k + \lambda_k \rho, t_k + \lambda_k^2 s), \quad \rho \geq -\rho_k, \quad s \in \mathbb{R},$$

solve the equations

$$\partial_t V_k - \partial_{\rho\rho} V_k = \frac{n-1}{\rho_k + \rho} \partial_\rho V_k + F(V_k)$$

and a subsequence of $\{V_k\}$ converges to nontrivial nonnegative solution of (9) with $n = 1$, which contradicts Theorem 3. Hence $\rho_\infty < \infty$. The functions

$$V_k(\rho, s) := \lambda_k^{2/(p-1)} \tilde{U}(\lambda_k \rho, t_k + \lambda_k^2 s), \quad \rho \geq 0, \quad s \in \mathbb{R},$$

solve the equations

$$\partial_t V_k - \partial_{\rho\rho} V_k = \frac{n-1}{\rho} \partial_\rho V_k + F(V_k)$$

and satisfy $|V_k(\rho_k, 0)| = 1$, $|V_k(\rho, s)| \leq 2^{2/(p-1)}$ for $|\rho - \rho_k| + \sqrt{|s|} \leq k$. Passing to a subsequence we may assume $V_k \rightarrow V$, where V is a nontrivial nonnegative bounded radial solution of (9). Replacing U by V we may assume that U is bounded.

Since U is bounded, the parabolic regularity implies that ∇U is bounded as well, hence

$$|U| + |\nabla U| \leq C. \quad (32)$$

Now we use similar doubling and scaling arguments as above to show the uniform decay estimate

$$|\tilde{U}(r, t)| r^{2/(p-1)} + |\nabla \tilde{U}(r, t)| r^{(p+1)/(p-1)} \leq C \quad (33)$$

(where the constant C is different from that in (32)). Assume on the contrary that there exist $r_k > 0$ and $t_k \in \mathbb{R}$ such that

$$|\tilde{U}(r_k, t_k)| r_k^{2/(p-1)} + |\nabla \tilde{U}(r_k, t_k)| r_k^{(p+1)/(p-1)} \rightarrow \infty.$$

Set

$$M(r, t) := |\tilde{U}(r, t)|^{(p-1)/2} + |\nabla \tilde{U}(r, t)|^{(p-1)/(p+1)}, \quad r > 0, \quad t \in \mathbb{R}$$

and $M_k := M(r_k, t_k)$. Then $M_k r_k \rightarrow \infty$ and passing to a subsequence we may assume $M_k > 2k/r_k$. The Doubling Lemma (with $X = [0, \infty) \times \mathbb{R}$, $D = (0, \infty) \times \mathbb{R}$ and $\text{dist} = \text{dist}_P$) guarantees that we may assume

$$M(r, t) \leq 2M_k \quad \text{whenever} \quad |r - r_k| + \sqrt{|t - t_k|} \leq \frac{k}{M_k}.$$

Set $\lambda_k := 1/M_k$ and

$$V_k(\rho, s) := \lambda_k^{2/(p-1)} \tilde{U}(r_k + \lambda_k \rho, t_k + \lambda_k^2 s).$$

Then

$$\begin{aligned} |V_k(0, 0)|^{(p-1)/2} + |\partial_\rho V_k(0, 0)|^{(p-1)/(p+1)} &= 1, \\ |V_k(\rho, s)|^{(p-1)/2} + |\partial_\rho V_k(\rho, s)|^{(p-1)/(p+1)} &\leq 2 \quad \text{whenever } |\rho| + \sqrt{|s|} \leq k, \end{aligned}$$

and V_k solves the equation

$$\partial_t V_k - \partial_{\rho\rho} V_k = \frac{n-1}{r_k/\lambda_k + \rho} \partial_\rho V_k + F(V_k).$$

Since $r_k/\lambda_k = r_k M_k \rightarrow \infty$, it is easy to pass to the limit to get a nontrivial nonnegative bounded solution V of (9) with $n = 1$. However, this contradicts Theorem 3. Consequently, (33) is true.

Next we use the energy functional

$$E(U(\cdot, t)) := \int_{\mathbb{R}^n} \left(\frac{1}{2} |\nabla U(x, t)|^2 - G(U(x, t)) \right) dx.$$

The arguments in [25, Example 51.28, the case $\lambda = 0$] guarantee that the system (9) is well posed in the space

$$\left. \begin{aligned} \mathcal{E} &:= \{W \in L^{p+1}(\mathbb{R}^n, \mathbb{R}^m) : \nabla W \in L^2(\mathbb{R}^n, \mathbb{R}^{mn})\}, \\ \|W\|_{\mathcal{E}} &:= \|W\|_{L^{p+1}} + \|\nabla W\|_{L^2} \end{aligned} \right\} \quad (34)$$

and the corresponding solution satisfies the energy identity

$$E(U(\cdot, t_2)) - E(U(\cdot, t_1)) = - \int_{t_1}^{t_2} \int_{\mathbb{R}^n} |U_t|^2(x, t) dx dt. \quad (35)$$

Estimates (33) and (32) guarantee $\|U(\cdot, t)\|_{\mathcal{E}} \leq C$ and $|E(u(\cdot, t))| \leq C$ with C independent of t , hence

$$\int_{\mathbb{R}} \int_{\mathbb{R}^n} |U_t|^2 dx dt < \infty$$

and

$$\int_{|t|>k} \int_{\mathbb{R}^n} |U_t|^2 dx dt \rightarrow 0 \quad \text{as } k \rightarrow \infty. \quad (36)$$

Next we claim

$$\sup_{x \in \mathbb{R}^n, |t|>2k} (|U(x, t)| + |\nabla U(x, t)|) \rightarrow 0 \quad \text{as } k \rightarrow \infty. \quad (37)$$

Assume on the contrary that there exist $x_k \in \mathbb{R}^n$ and $t_k \in \mathbb{R}$, $|t_k| > 2k$, such that

$$|U(x_k, t_k)| + |\nabla U(x_k, t_k)| \geq c_0 > 0.$$

Estimate (33) shows that the sequence $\{x_k\}$ is bounded so that we may assume $x_k \rightarrow x_\infty$. Set $V_k(x, t) := U(x, t - t_k)$. Then a subsequence of $\{V_k\}$ converges (locally uniformly in C^1) to a nonnegative radial solution V of (9). Estimate $|V(x_\infty, 0)| + |\nabla V(x_\infty, 0)| \geq c_0$ shows that V is nontrivial and estimate (36) guarantees that V does not depend on t . However, this contradicts the elliptic Liouville theorem [28, Proposition 5(i)].

Estimates (33) and (37) guarantee $E(U(\cdot, t)) \rightarrow 0$ as $|t| \rightarrow \infty$, so that $E(U(\cdot, t)) \equiv 0$ by the monotonicity of $t \mapsto E(U(\cdot, t))$. Consequently, $U_t \equiv 0$ which contradicts [28, Proposition 5(i)]. \square

3 Proofs of Theorems 5, 6 and 7

Proof of Theorem 5. The proof will follow that of Theorem 1 but we will also need some additional arguments.

Assume on the contrary that there exists a positive bounded solution u of (15). By using doubling and scaling arguments we first show that we may assume

$$u(x, t) + |\nabla u(x, t)| \leq C \quad \text{for all } x \in \overline{\mathbb{R}_+^n}, \quad t \in \mathbb{R}. \quad (38)$$

Assume that (38) fails. Since $u \leq C_u$ for some $C_u > 0$, we can find (x_k, t_k) such that $|\nabla u(x_k, t_k)| \rightarrow \infty$. Set

$$M(x, t) := u^{q-1}(x, t) + |\nabla u(x, t)|^{(q-1)/q},$$

$M_k := M(x_k, t_k)$ and $\lambda_k := 1/M_k$. The Doubling Lemma (with $X = D = \overline{\mathbb{R}_+^n} \times \mathbb{R}$ and $\text{dist} = \text{dist}_P$) guarantees that we may assume

$$M(x, t) \leq 2M_k \quad \text{whenever} \quad |x - x_k| + \sqrt{|t - t_k|} \leq \frac{k}{M_k}.$$

Passing to a subsequence we may assume $c_k := x_{k,1}M_k \rightarrow c_\infty \in [0, \infty]$, where $x_{k,1}$ denotes the first component of x_k . If $c_\infty = \infty$ then setting

$$v_k(y, s) := \lambda_k^{1/(q-1)} u(x_k + \lambda_k y, t_k + \lambda_k^2 s), \quad y \in \mathbb{R}^n, \quad y_1 \geq -c_k, \quad s \in \mathbb{R},$$

a suitable subsequence of $\{v_k\}$ converges to a nonnegative bounded solution v of the linear heat equation in $\mathbb{R}^n \times \mathbb{R}$ satisfying $|\nabla v(0, 0)| = 1$, which contradicts the Liouville theorem for the linear heat equation (see [7, Theorem 1] or [6, Theorem 4] and cf. also [20]). Therefore we have $c_\infty < \infty$. Set $x_k^0 := (0, x_{k,2}, \dots, x_{k,n})$, $y_k := (c_k, 0, 0, \dots, 0)$ and

$$v_k(y, s) := \lambda_k^{1/(q-1)} u(x_k^0 + \lambda_k y, t_k + \lambda_k^2 s), \quad y \in \mathbb{R}_+^n, \quad s \in \mathbb{R}.$$

Then $v_k^{q-1}(y_k, 0) + |\nabla v_k(y_k, 0)|^{(q-1)/q} = 1$ and a suitable subsequence of $\{v_k\}$ converges to a nonnegative nontrivial (hence positive) bounded solution v of (15) with bounded spatial derivatives. Replacing u by v we obtain (38).

If (38) is true then the function

$$v(y, s) = \lambda^{1/(q-1)} u(\lambda y, \lambda^2 s) \quad \text{where} \quad \lambda^{1/(q-1)} = 1/C$$

is a positive solution of (15) satisfying (38) with $C = 1$. Hence, replacing u by v we may assume

$$u(x, t) + |\nabla u(x, t)| \leq 1 \quad \text{for all } x \in \overline{\mathbb{R}_+^n}, \quad t \in \mathbb{R}. \quad (39)$$

Next we prove that

$$u_{x_1}(x, t) \leq 0 \quad \text{for all } x \in \mathbb{R}_+^n, \quad t \in \mathbb{R}. \quad (40)$$

The function $z := u_{x_1}$ is bounded and satisfies

$$z_t - \Delta z = 0 \quad \text{in } \mathbb{R}_+^n \times \mathbb{R}, \quad z < 0 \quad \text{on } \partial \mathbb{R}_+^n \times \mathbb{R}.$$

In order to prove (40) it is sufficient to show $z(x, t) \leq \varepsilon x_1$ for any $\varepsilon > 0$. Fix $\varepsilon > 0$ and set $v(x, t) := z(x, t) - \varepsilon x_1$. Since z is bounded, there exists $\lambda = \lambda(\varepsilon) > 0$ such that $v(x, t) < 0$ for $x_1 \geq \lambda$. To show $v(x, t) \leq 0$ for $x_1 < \lambda$ we will proceed similarly as in the proof of [24, Theorem 2.4].

Denoting $T_\lambda := \{x \in \mathbb{R}^n : 0 < x_1 < \lambda\}$ the function v satisfies

$$v_t - \Delta v = 0 \quad \text{in } T_\lambda \times \mathbb{R}, \quad v(x, t) < 0 \quad \text{on } \partial T_\lambda \times \mathbb{R}.$$

Choosing $q \in (0, \pi^2/\lambda^2)$, [4] guarantees the existence of a smooth positive function h on $\overline{T_\lambda}$ such that

$$\Delta h + qh = 0 \quad \text{in } T_\lambda \quad \text{and} \quad h(x) \rightarrow +\infty \quad \text{as } |x| \rightarrow \infty, \quad x \in \overline{T_\lambda}.$$

In particular $h(x) \geq h_0 > 0$. Set $w := e^{qt}v/h$. Then w satisfies

$$w_t - \Delta w - \frac{2\nabla h}{h} \cdot \nabla w = 0 \quad \text{in } T_\lambda \times \mathbb{R}, \quad w \leq 0 \quad \text{on } \partial T_\lambda \times \mathbb{R}.$$

Fix $t_0 < t_1$ and consider $(x, t) \in \overline{T_\lambda} \times [t_0, t_1]$. Then $w(x, t) \rightarrow 0$ as $|x| \rightarrow \infty$ and the maximum principle guarantees

$$\sup_{x \in T_\lambda} w^-(x, t_1) \leq \sup_{x \in T_\lambda} w^-(x, t_0),$$

where $w^-(x, t) := -\min(w(x, t), 0)$. For v the above inequality means

$$\sup_{x \in T_\lambda} \frac{v^-(x, t_1)}{h(x)} \leq e^{-q(t_1-t_0)} \sup_{x \in T_\lambda} \frac{v^-(x, t_0)}{h(x)}.$$

In view of boundedness of v on $T_\lambda \times \mathbb{R}$, letting $t_0 \rightarrow -\infty$ we obtain that $v(x, t_1) \leq 0$. This concludes the proof of (40).

Denote $c_0 := u(0, 0)$ and $\tilde{\beta} := 1/2(q-1)$. For $y \in \mathbb{R}_+^n$, $s \in \mathbb{R}$ and $k = 1, 2, \dots$ set

$$w_k(y, s) := (k-t)^{\tilde{\beta}} u(y\sqrt{k-t}, t), \quad \text{where } s = -\log(k-t), \quad t < k.$$

Set also $s_k := -\log k$ and notice that $w = w_k$ solve the problem

$$\left. \begin{aligned} w_s &= \Delta w - \frac{1}{2}y \cdot \nabla w - \tilde{\beta}w = \frac{1}{\rho} \nabla \cdot (\rho \nabla w) - \tilde{\beta}w && \text{in } \mathbb{R}_+^n \times \mathbb{R}, \\ w_\nu &= w^q && \text{on } \partial \mathbb{R}_+^n \times \mathbb{R}, \end{aligned} \right\} \quad (41)$$

where $\rho(y) = e^{-|y|^2/4}$. In addition,

$$w_k(0, s_k) = k^{\tilde{\beta}} c_0, \quad \|w_k(\cdot, s)\|_\infty \leq e^{2\tilde{\beta}} k^{\tilde{\beta}} \quad \text{for } s \in [s_k - 2, \infty).$$

Set

$$E_k(s) := \frac{1}{2} \int_{\mathbb{R}_+^n} (|\nabla w_k(y, s)|^2 + \tilde{\beta} w_k^2(y, s)) \rho(y) dy - \frac{1}{q+1} \int_{\partial \mathbb{R}_+^n} w_k^{q+1}(\xi, s) \rho(\xi) dS_\xi.$$

Then $E_k(s) \geq 0$ (see [3]) and, given $\sigma < s_k$, we also have

$$\begin{aligned} & \frac{1}{2} \left(\int_{\mathbb{R}_+^n} w_k^2(y, s_k) \rho(y) dy - \int_{\mathbb{R}_+^n} w_k^2(y, \sigma) \rho(y) dy \right) \\ &= -2 \int_{\sigma}^{s_k} E_k(s) ds + \frac{q-1}{q+1} \int_{\sigma}^{s_k} \int_{\partial \mathbb{R}_+^n} w_k^{q+1}(\xi, s) \rho(\xi) dS_{\xi} ds, \\ & \int_{\sigma}^{s_k} \int_{\mathbb{R}_+^n} \left| \frac{\partial w_k}{\partial s}(y, s) \right|^2 \rho(y) dy ds = E_k(\sigma) - E_k(s_k) \leq E_k(\sigma). \end{aligned}$$

Since (40) guarantees $\partial w_k / \partial y_1 \leq 0$, we have

$$\begin{aligned} \sqrt{\pi/2} \int_{\partial \mathbb{R}_+^n} w_k(\xi, s) \rho(\xi) dS_{\xi} &= \int_{\mathbb{R}_+^n} w_k((0, y_2, y_3, \dots, y_n), s) \rho(y) dy \\ &\geq \int_{\mathbb{R}_+^n} w_k(y, s) \rho(y) dy. \end{aligned}$$

Consequently, multiplying the equation in (41) by ρ and integrating over $y \in \mathbb{R}_+^n$ yields

$$\begin{aligned} \frac{d}{ds} \int_{\mathbb{R}_+^n} w_k(y, s) \rho(y) dy + \tilde{\beta} \int_{\mathbb{R}_+^n} w_k(y, s) \rho(y) dy &= \int_{\partial \mathbb{R}_+^n} w_k^q(\xi, s) \rho(\xi) dS_{\xi} \\ &\geq C_{n-1,q} \left(\int_{\partial \mathbb{R}_+^n} w_k(\xi, s) \rho(\xi) dS_{\xi} \right)^q \geq \hat{C}_{n-1,q} \left(\int_{\mathbb{R}_+^n} w_k(y, s) \rho(y) dy \right)^q, \end{aligned}$$

which again implies the estimates of the type

$$\begin{aligned} \int_{\mathbb{R}_+^n} w_k(y, s) \rho(y) dy &\leq \tilde{C}_{n-1,q}, \\ \int_{\sigma}^{s_k} \int_{\partial \mathbb{R}_+^n} w_k^q(\xi, s) \rho(\xi) dS_{\xi} ds &\leq \tilde{C}_{n-1,q} (1 + \tilde{\beta}(s_k - \sigma)). \end{aligned}$$

In the same way as in the proof of Theorem 1, the estimates above guarantee $E_k(s_k - 1) \leq \tilde{C}(n-1, q) k^{\tilde{\beta}}$ for suitable $\tilde{C}(n-1, q)$ and, consequently,

$$\int_{s_k-1}^{s_k} \int_{\mathbb{R}_+^n} \left| \frac{\partial w_k}{\partial s}(y, s) \right|^2 \rho(y) dy ds \leq \tilde{C}(n-1, q) k^{\tilde{\beta}}. \quad (42)$$

Denote $\lambda_k := k^{-1/2}$ and set

$$v_k(z, \tau) := \lambda_k^{1/(q-1)} w_k(\lambda_k z, \lambda_k^2 \tau + s_k), \quad z \in \mathbb{R}_+^n, \quad -k \leq \tau \leq 0.$$

Then $0 < v_k \leq e^{2\tilde{\beta}}$, $v_k(0, 0) = c_0$,

$$\begin{aligned} \frac{\partial v_k}{\partial \tau} - \Delta v_k &= -\lambda_k^2 \left(\frac{1}{2} z \cdot \nabla v_k + \tilde{\beta} v_k \right) && \text{in } \mathbb{R}_+^n \times (-k, 0), \\ v_{\nu} &= v^q && \text{on } \partial \mathbb{R}_+^n \times (-k, 0), \end{aligned}$$

and, denoting $\tilde{\alpha} := -n + 2 + 2/(q-1)$ and using (42) we also have

$$\begin{aligned} \int_{-k}^0 \int_{|z| < \sqrt{k}} \left| \frac{\partial v_k}{\partial \tau}(z, \tau) \right|^2 dz d\tau &= \lambda_k^{\tilde{\alpha}} \int_{s_k-1}^{s_k} \int_{|y| < 1} \left| \frac{\partial w_k}{\partial s}(y, s) \right|^2 dy ds \\ &\leq C(q) e^{1/4} k^{-\tilde{\alpha}/2 + \tilde{\beta}} \rightarrow 0 \quad \text{as } k \rightarrow \infty. \end{aligned}$$

As in the proof of Theorem 1 (cf. also [3]), a subsequence of $\{v_k\}$ converges to a positive solution $v = v(z)$ of the problem $\Delta v = 0$ in \mathbb{R}_+^n , $v_\nu = v^q$ on $\partial\mathbb{R}_+^n$, which contradicts the elliptic Liouville theorem in [15]. \square

Proof of Theorem 6. Due to Theorem 5 we may assume $n > 2$ and $n-1 \leq q(n-2) < n$. Assume that u is a positive classical bounded solution of (15) satisfying (16). Similarly as in the proof of Theorem 5 we will first show that we may assume

$$u + |\nabla u| \leq C \quad (43)$$

and then (similarly as in the proof of Theorem 4) we will prove that u satisfies suitable decay estimates which allow us to use the energy functional

$$E(\varphi) := \frac{1}{2} \int_{\mathbb{R}_+^n} |\nabla \varphi(x)|^2 dx - \frac{1}{q+1} \int_{\mathbb{R}^{n-1}} \varphi(0, \tilde{x})^{q+1} d\tilde{x} \quad (44)$$

to show that u is time independent.

Assume that (43) fails. Since $u \leq C_u$ for some $C_u > 0$, we can find (x_k, t_k) such that $|\nabla u(x_k, t_k)| \rightarrow \infty$. Set

$$M(x, t) := u^{q-1}(x, t) + |\nabla u(x, t)|^{(q-1)/q},$$

$M_k := M(x_k, t_k)$ and $\lambda_k := 1/M_k$. In the same way as in the proof of Theorem 5, the Doubling Lemma guarantees that we may assume

$$M(x, t) \leq 2M_k \quad \text{whenever} \quad |x - x_k| + \sqrt{|t - t_k|} \leq \frac{k}{M_k} \quad (45)$$

and then the Liouville theorem for the linear heat equation [7, Theorem 1] implies that we may assume $c_k := x_{k,1} M_k \rightarrow c_\infty \in [0, \infty)$.

Assumption (16) guarantees $u(x_1, \tilde{x}, t) = v(x_1, r, t)$, where $r = |\tilde{x}|$. Passing to a subsequence we may assume $\rho_k := r_k/\lambda_k \rightarrow \rho_\infty \in [0, \infty]$, where $r_k = |\tilde{x}_k|$. If $\rho_\infty = \infty$ then we set

$$w_k(y, \rho, s) := \lambda_k^{1/(q-1)} v(\lambda_k y, r_k + \lambda_k \rho, t_k + \lambda_k^2 s), \quad y \geq 0, \quad \rho \geq -\rho_k, \quad s \in \mathbb{R}.$$

Then

$$\begin{aligned} w_k^{q-1}(c_k, 0, 0) + |\nabla w_k(c_k, 0, 0)|^{(q-1)/q} &= 1, \\ w_k^{q-1} + |\nabla w_k| &\leq 2, \quad \text{whenever} \quad \sqrt{(y - c_k)^2 + \rho^2} + \sqrt{|s|} \leq k \end{aligned}$$

and w_k satisfy the equation

$$w_s - w_{yy} - w_{\rho\rho} = \frac{n-2}{\rho_k + \rho} w_\rho, \quad y > 0, \rho > -\rho_k \quad s \in \mathbb{R},$$

and the boundary condition $w_y = -w^q$ for $y = 0$. Consequently, a subsequence of $\{w_k\}$ converges to a nonnegative nontrivial solution of (15) with $n = 2$ which contradicts Theorem 5. Hence $\rho_\infty < \infty$. Set

$$v_k(y, s) := \lambda_k^{1/(q-1)} u(\lambda_k y, t_k + \lambda_k^2 s), \quad y \in \mathbb{R}_+^n, \quad s \in \mathbb{R},$$

fix $\tilde{y} \in \mathbb{R}^{n-1}$ with $|\tilde{y}| = 1$ and set $y_k = (c_k, \rho_k \tilde{y})$. Then v_k are solutions of (15) satisfying (16), $v_k^{(q-1)}(y_k, 0) + |\nabla v_k(y_k, 0)|^{(q-1)/q} = 1$ and the bound (45) guarantees that a suitable subsequence of $\{v_k\}$ converges to a positive bounded solution v of (15) satisfying (16) and having bounded spatial derivatives. Replacing u by v we obtain (43).

Next we use doubling and scaling arguments together with Theorem 5 in order to show

$$u(0, \tilde{x}, t) \leq C |\tilde{x}|^{-1/(q-1)} \quad \text{for all } \tilde{x} \in \mathbb{R}^{n-1}. \quad (46)$$

Notice that the monotonicity property (40) will then guarantee

$$u(x, t) \leq C |\tilde{x}|^{-1/(q-1)} \quad \text{for all } \tilde{x} \in \mathbb{R}_+^n. \quad (47)$$

Assume on the contrary that (46) fails. Then there exist \tilde{x}_k, t_k such that

$$u(0, \tilde{x}_k, t_k) |\tilde{x}_k|^{1/(q-1)} \rightarrow \infty.$$

Due to (43) we have $|\tilde{x}_k| \rightarrow \infty$. Denote $r = |\tilde{x}|$, $v(x_1, r, t) = u(x_1, \tilde{x}, t)$, $r_k = |\tilde{x}_k|$ and $M(r, t) = v(0, r, t)^{q-1}$ for $(r, t) \in (0, \infty) \times \mathbb{R}$. Then $M(r_k, t_k) r_k \rightarrow \infty$ so that we may assume $M_k := M(r_k, t_k) > 2k/r_k$. Now the Doubling Lemma (with $X = [0, \infty) \times \mathbb{R}$, $D = (0, \infty) \times \mathbb{R}$ and $\text{dist} = \text{dist}_P$) guarantees that we may also assume

$$M(r, t) \leq 2M_k \quad \text{whenever } |r - r_k| + \sqrt{|t - t_k|} \leq \frac{k}{M_k}. \quad (48)$$

Set $\lambda_k = 1/M_k$ and

$$w_k(y, \rho, s) := \lambda_k^{1/(q-1)} v(\lambda_k y, r_k + \lambda_k \rho, t_k + \lambda_k^2 s), \quad y \geq 0, \rho \geq -r_k/\lambda_k, \quad s \in \mathbb{R}.$$

Then $w(0, 0, 0) = 1$ and (48), (40) guarantee $w_k \leq 2^{1/(q-1)}$ whenever $|\rho| + \sqrt{|s|} \leq k$. In addition $w = w_k$ is a positive solution of the equation

$$w_s - w_{yy} - w_{\rho\rho} = \frac{n-2}{r_k/\lambda_k + \rho} w_\rho$$

complemented by the boundary condition $w_y = -w^q$ for $y = 0$. Since $r_k/\lambda_k \rightarrow \infty$, it is easy to pass to the limit (in the weak formulation of the problem) to obtain a positive

bounded solution of the problem $w_t - \Delta w = 0$ in $\mathbb{R}_+^2 \times \mathbb{R}$, $w_\nu = w^q$ on $\partial\mathbb{R}_+^2 \times \mathbb{R}$, which contradicts Theorem 5. Consequently, (46) and (47) are true.

To prove the decay of u with respect to x_1 we use the representation formula

$$\left. \begin{aligned} u(x, t) &= \int_{\mathbb{R}_+^n} G(x, y, t - T) u(y, T) dy \\ &+ \int_T^t \int_{\mathbb{R}^{n-1}} \partial_{y_1} G(x, (0, \tilde{y}), t - s) u((0, \tilde{y}), s) d\tilde{y} ds =: A_1 + A_2, \end{aligned} \right\} \quad (49)$$

for $x_1 > 0$ and $t > T$, where

$$G(x, y, t) = \frac{1}{(4\pi t)^{n/2}} (e^{-|x-y|^2/4t} - e^{-|x'-y|^2/4t}), \quad x' := (-x_1, \tilde{x}).$$

Notice that

$$\begin{aligned} 0 &\leq G(x, y, t) \leq C t^{-n/2} e^{-|x-y|^2/4t}, \\ 0 &\leq \partial_{y_1} G(x, (0, \tilde{y}), t) \leq C x_1 t^{-n/2-1} e^{-|x-y|^2/4t}. \end{aligned}$$

Introducing the new variable $z = (x - y)/2\sqrt{t - T}$ in A_1 we have

$$A_1 \leq C \int_{\{z: z_1 \leq x_1/2\sqrt{t-T}\}} e^{-|z|^2} u(x - 2z\sqrt{t-T}, T) dz \rightarrow 0 \quad \text{as } T \rightarrow -\infty,$$

due to the Lebesgue dominated convergence theorem and the pointwise convergence $u(x - 2z\sqrt{t-T}, T) \rightarrow 0$ for $\tilde{z} \neq 0$ (which follows from (47)). Using estimate (47) we also have

$$A_2 \leq C \int_T^t x_1 (t-s)^{-3/2} e^{-x_1^2/4(t-s)} I(\tilde{x}, t-s) ds,$$

where

$$I(\tilde{x}, t) := \int_{\mathbb{R}^{n-1}} (4t)^{-(n-1)/2} e^{-|\tilde{x}-\tilde{y}|^2/4t} |\tilde{y}|^{-1/(q-1)} d\tilde{y}.$$

Introducing the variable $\tilde{z} = (\tilde{x} - \tilde{y})/2\sqrt{t}$ we obtain

$$\begin{aligned} I(\tilde{x}, t) &= \int_{\mathbb{R}^{n-1}} e^{-|\tilde{z}|^2} |\tilde{x} - 2\sqrt{t}\tilde{z}|^{-1/(q-1)} d\tilde{z} \\ &\leq \int_{|\tilde{x}-2\sqrt{t}\tilde{z}| > \sqrt{t}} e^{-|\tilde{z}|^2} t^{-1/2(q-1)} d\tilde{z} + \int_{|\tilde{x}-2\sqrt{t}\tilde{z}| < \sqrt{t}} |\tilde{x} - 2\sqrt{t}\tilde{z}|^{-1/(q-1)} d\tilde{z} \\ &\leq C_1 t^{-1/2(q-1)} + C_2 \int_0^{1/2} (r\sqrt{t})^{-1/(q-1)} \cdot r^{n-2} dr \\ &\leq C_3 t^{-1/2(q-1)}. \end{aligned}$$

Consequently,

$$A_2 \leq C \int_{-\infty}^t x_1 (t-s)^{-3/2} (t-s)^{-1/2(q-1)} e^{-x_1^2/4(t-s)} ds$$

and introducing the new variable τ satisfying $t - \tau = (t - s)/x_1^2$ we obtain

$$A_2 \leq Cx_1^{-1/(q-1)} \int_{-\infty}^t (t - \tau)^{-3/2-1/2(q-1)} e^{-1/4(t-\tau)} d\tau = C(q)x_1^{-1/(q-1)}.$$

Since the last estimate of A_2 does not depend on T and $A_1 \rightarrow 0$ as $T \rightarrow -\infty$, (49) and (47) imply

$$u(x, t) \leq C|x|^{-1/(q-1)} \quad \text{for all } (x, t) \in \mathbb{R}_+^n \times \mathbb{R}. \quad (50)$$

Next we use doubling and scaling arguments again to prove the estimate

$$|\nabla u(x, t)| \leq C|x|^{-q/(q-1)} \quad \text{for all } (x, t) \in \mathbb{R}_+^n \times \mathbb{R}. \quad (51)$$

Assume on the contrary that there exist x_k, t_k such that

$$|\nabla u(x_k, t_k)| \cdot |x_k|^{q/(q-1)} \rightarrow \infty.$$

Due to (43) we have $|x_k| \rightarrow \infty$. Set

$$M(x, t) := u(x, t)^{q-1} + |\nabla u(x, t)|^{(q-1)/q}.$$

Then without loss of generality we may assume $M_k := M(x_k, t_k) > 2k/|x_k|$. The Doubling Lemma (with $X = \overline{\mathbb{R}_+^n} \times \mathbb{R}$, $D = (\overline{\mathbb{R}_+^n} \setminus \{0\}) \times \mathbb{R}$ and $\text{dist} = \text{dist}_P$) shows that we may assume

$$M(x, t) \leq 2M_k \quad \text{whenever } |x - x_k| + \sqrt{|t - t_k|} \leq \frac{k}{M_k}.$$

Finally, we may also assume that $c_k := x_{k,1}M_k \rightarrow c_\infty \in [0, \infty]$, where $x_{k,1}$ denotes the first component of x_k . Set $\lambda_k = 1/M_k$ and

$$v_k(y, s) := \lambda_k^{1/(q-1)} u(x_k + \lambda_k y, t_k + \lambda_k^2 s), \quad y \in \mathbb{R}^n, \quad y_1 \geq -c_k, \quad s \in \mathbb{R}.$$

Then v_k solves the linear heat equation and satisfies the boundary condition $v_\nu = v^q$. Since (50) implies $u(x, t)^{q-1} \leq C/|x|$, we have $|\nabla u(x_k, t_k)|^{(q-1)/q} > M_k/2$ for k large enough, hence $|\nabla v_k(0, 0)| > 2^{-q/(q-1)}$. On the other hand,

$$v_k(0, 0) \leq C(M_k|x_k|)^{-1/(q-1)} \rightarrow 0$$

and

$$v_k^{q-1} + |\nabla v_k|^{(q-1)/q} \leq 2 \quad \text{for } |y| + \sqrt{|s|} \leq k, \quad y_1 \geq -x_{k,1}/\lambda_k.$$

If $c_\infty = \infty$ then a suitable subsequence of v_k converges to the nonnegative solution v of the linear heat equation in $\mathbb{R}^n \times \mathbb{R}$, $v(0, 0) = 0$ and $\nabla v(0, 0) \neq 0$ which contradicts [7, Theorem 1]. If $c_\infty < \infty$ then a subsequence of v_k converges to the nonnegative solution v of the linear heat equation in $\{y \in \mathbb{R}^n : y_1 > -c_\infty\} \times \mathbb{R}$ satisfying the boundary condition $v_\nu = v^q$ and $v(0, 0) = 0$, $\nabla v(0, 0) \neq 0$, which yields a contradiction again. Consequently, (51) is true.

Estimates (50) and (51) guarantee that the energy $E(u(\cdot, t))$ is well defined and that we can use the same arguments as in the proof of Theorem 4 to show $E(u(\cdot, t)) \equiv 0$.

Consequently, u is time-independent which contradicts the elliptic Liouville theorem in [15]. \square

Proof of Theorem 7. Set

$$M(t) := \max_{x \in \overline{\Omega}, \tau \in [T/4, t]} (u(x, \tau)^{q-1} + |\nabla u(x, \tau)|^{(q-1)/q}), \quad t \in [T/4, T].$$

We will prove $M(t)\sqrt{T-t} \leq C = C(u)$ for $t \in (T/2, T)$.

Assume on the contrary that there exist $t_k \in (T/2, T)$ such that

$$M(t_k)\sqrt{T-t_k} \rightarrow \infty.$$

We may assume $M(t_k)\sqrt{T-t_k} > 2k$ and $M(t_k) > 2M(T/2)$. Using the Doubling Lemma (with $X = [T/4, T]$, $D = [T/4, T)$ and $\text{dist}(t, \tilde{t}) = \sqrt{|t - \tilde{t}|}$) we find $\tilde{t}_k \in [T/4, T)$ such that

$$M_k := M(\tilde{t}_k) \geq M(t_k), \quad M_k\sqrt{T-\tilde{t}_k} > 2k \quad (52)$$

and

$$M(t) \leq 2M_k \quad \text{for all } t \in [T/4, T), \quad \sqrt{|t - \tilde{t}_k|} \leq \frac{k}{M_k}. \quad (53)$$

In fact, the monotonicity of M and (53) guarantee

$$M(t) \leq 2M_k \quad \text{for all } t \in \left[T/4, \tilde{t}_k + \frac{k^2}{M_k^2}\right). \quad (54)$$

Inequalities $M_k \geq M(t_k) > 2M(T/2)$ guarantee $\tilde{t}_k > T/2$. Fix $x_k \in \overline{\Omega}$ and $\tau_k \in [T/4, \tilde{t}_k]$ such that

$$M_k = u(x_k, \tau_k)^{q-1} + |\nabla u(x_k, \tau_k)|^{(q-1)/q}$$

and notice that $M(\tau_k) = M_k$. Next we distinguish two cases:

- (i) $u(x_k, \tau_k)^{q-1} > \frac{1}{2}M_k$,
- (ii) $u(x_k, \tau_k)^{q-1} \leq \frac{1}{2}M_k$.

Case (i): Since $u(x_k, \tau_k)^{q-1} > M_k/2 > M(T/2)$, we have $u(x_k, \tau_k) > \max\{u(x, t) : x \in \overline{\Omega}, t \in [T/4, T/2]\}$ and the maximum principle guarantees that there exist $\hat{x}_k \in \partial\Omega$ and $\hat{\tau}_k \in (T/2, \tau_k] \subset [T/4, \tilde{t}_k]$ such that

$$u(\hat{x}_k, \hat{\tau}_k) = \max\{u(x, t) : x \in \overline{\Omega}, t \in [T/4, \tau_k]\} \geq u(x_k, \tau_k).$$

Consequently, $u^{q-1}(\hat{x}_k, \hat{\tau}_k) > M_k/2$. Set

$$\lambda_k := 1/M_k, \quad \Omega_k := \{y \in \mathbb{R}^n : \hat{x}_k + \lambda_k R_k y \in \Omega\},$$

where R_k is a rotation operator such that $(-1, 0, 0, \dots, 0)$ is the exterior normal vector of $\partial\Omega_k$ at 0. Given $y \in \overline{\Omega}_k$ and $s \in I_k := \{s : \hat{\tau}_k + \lambda_k^2 s \in [T/4, T)\}$, set also

$$v_k(y, s) := \lambda_k^{1/(q-1)} u(\hat{x}_k + \lambda_k R_k y, \hat{\tau}_k + \lambda_k^2 s). \quad (55)$$

Then v_k solve the equation and the boundary condition in (17) in $\Omega_k \times I_k$ and on $\partial\Omega_k \times I_k$, respectively, $v_k^{q-1}(0, 0) > 1/2$, $v_k^{q-1}(y, s) + |\nabla v_k(y, s)|^{(q-1)/q} \leq 2$ for all $(y, s) \in \overline{\Omega}_k \times I_k$ satisfying $|s| \leq k^2$. The arguments in [16] guarantee that a subsequence of $\{v_k\}$ converges in C_{loc}^1 to a positive entire solution of (15) which contradicts Theorem 5.

Case (ii): Denote $d_k := \text{dist}(x_k, \partial\Omega)$ and choose $\tilde{x}_k \in \partial\Omega$ such that $|\tilde{x}_k - x_k| = d_k$. Set also $\hat{\tau}_k := \tau_k$ and $c_k := d_k M_k$. We may assume $c_k \rightarrow c_\infty \in [0, \infty]$.

If $c_\infty < \infty$ then we set $\hat{x}_k := \tilde{x}_k$, $y_k := (c_k, 0, 0, \dots, 0)$ and define $\lambda_k, \Omega_k, R_k, I_k, v_k$ as in Case (i). Notice that $R_k y_k = \tilde{x}_k - \hat{x}_k$. Similarly as in Case (i), v_k solve the equation and the boundary condition in (17) in $\Omega_k \times I_k$ and on $\partial\Omega_k \times I_k$, respectively, $v_k^{q-1}(y_k, 0) + |\nabla v_k(y_k, 0)|^{(q-1)/q} = 1$, $v_k^{q-1}(y, s) + |\nabla v_k(y, s)|^{(q-1)/q} \leq 2$ for all $(y, s) \in \overline{\Omega}_k \times I_k$ satisfying $|s| \leq k^2$ and the arguments in [16] guarantee that a subsequence of $\{v_k\}$ converges in C_{loc}^1 to a positive entire solution of (15) which contradicts Theorem 5.

If $c_\infty = \infty$ then we set $\hat{x}_k := x_k$, define R_k as the identity and $\lambda_k, \Omega_k, I_k, v_k$ as in Case (i). Now a subsequence of $\{v_k\}$ converges in C_{loc}^1 to a nonnegative bounded solution of the linear heat equation in $\mathbb{R}^n \times \mathbb{R}$. Since

$$|\nabla u(\hat{x}_k, \hat{\tau}_k)|^{(q-1)/q} = |\nabla u(x_k, \tau_k)|^{(q-1)/q} \geq M_k/2$$

we have $|\nabla v_k(0, 0)| \geq 1/2$, hence v is nonconstant which contradicts the Liouville theorem for the linear heat equation [7, Theorem 1]. \square

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